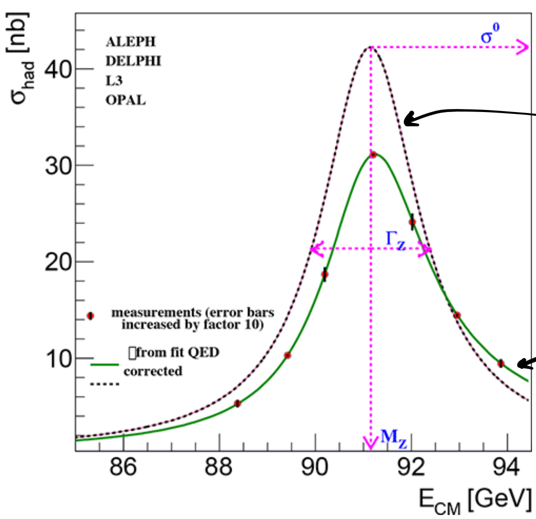
QED radiation

- Andrea Wulzer's GGI Lectures on Collider Physics
- Peskin 17.5

We have seen how the study of the Z boson resonance allows for a very precise determination of its properties.

However, if we compare the result we obtained with data, we get a huge discrepancy:

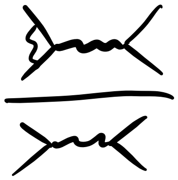
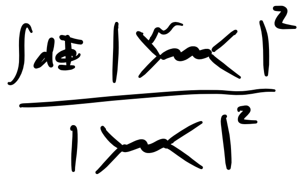


Z-boson resonance as computed by just resumming the Z-prop.

data

Why our calculation is so wrong?

We neglected higher order corrections. But these are expected to be

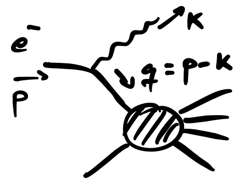
 $\sim \dfrac{e^2}{16\pi^2}$  $\sim \dfrac{e^2}{16\pi^2}$

so $\sim 10^{-3}$, should be small!

It turns out that this estimate is wrong, and radiative corrections may be larger.

- IR enhancement

Consider the e^- emitting a photon before entering into the "hard" process



in our case, the hard part is $e^+e^- \to \mu^+\mu^-$, but the discussion is general.

Fermion propagator contains a factor of $1/(q^2 - m_e^2)$.

Let us use lightcone coordinates. We write

$$p^\mu = (p^0, p^1, p^2, p^3)$$

$$\hookrightarrow \quad p^+ = p^0 + p^3 \,, \quad p^- = p^0 - p^3 \,, \quad \vec{p}_\perp = (p_1, p_2)$$

we have

$$p \cdot q = \frac{1}{2} p^+ q^- + \frac{1}{2} p^- q^+ - \vec{p}_\perp \cdot \vec{q}_\perp$$

The momenta of the incoming e^- & photon in these coordinates is

$$p = \left(p^+ = E + p^3 \,, \; p^- = \frac{m_e^2}{p^+} \,, \; \vec{p}_\perp = 0 \right)$$

$$k = \left(k^+ = (1-x) p^+ \,, \; k^- = \frac{|\vec{p}_\perp|^2}{k^+} \,, \; \vec{p}_\perp \right)$$

From this,

$$q = \left(x p^+ , \; \frac{m^2}{p^+} - \frac{p_\perp^2}{(1-x) p^+} \,, -\vec{p}_\perp \right)$$

Since the incoming e^- and outgoing photon are on-shell, the "internal" electron is necessarily off-shell. This is general,

$$p_3^2 = (p_1 - p_2)^2 = m_1^2 - 2 p_1 \cdot p_2$$

so $p_3^2 \to m^3$ only if
- $p_1 \cdot p_2 = 0$ (collinear)
- $p_2 \to 0$ (soft)

Therefore, whenever the virtuality of the internal electron is small, namely for collinear and soft radiation, the propagator is very large and these are configurations that 1) dominate and 2) the enhancement violates the naive counting we had before, since we were only considering a power counting in α, while now we see that kinematic configurations may be enhanced by big ratios of scales.

In terms of the previous variables, the virtuality is

$$q^2 - m_e^2 = - \frac{1+x}{1-x} |\vec{p}_\perp|^2 - (1-x)\, m_e^2$$

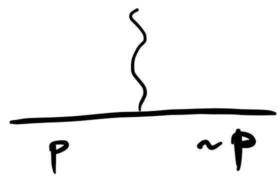
- soft photon : $p_\perp \sim \varepsilon p^+$ $\Rightarrow$ virt $\sim \varepsilon$
  $1-x \sim \varepsilon$

- Collinear photon : $x = \mathcal{O}(1)$ $\Rightarrow$ virt $\sim m_e$
  $p_\perp \sim \varepsilon p_\perp$

For a massive electron, in the collinear region the virtuality is always of order m_e.

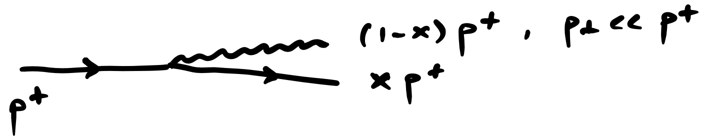
Still, if the typical scale of the process is $s \gg m_e^2$, this is still a small ratio.

In our case, the soft region is not particularly interesting since



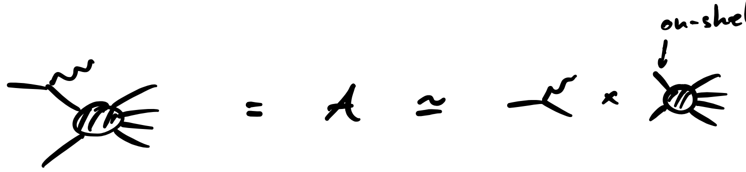
it will not change the kinematics at LO.

But the collinear region



the energy that enters in the hard process is $\mathcal{O}(1)$ different.

- The Feynman amplitude for low virtuality photon emissions has a factorized and universal form



$$= A_{split} \times A_{hard} \times \frac{1}{q^2 - m_e^2}$$

The splitting "amplitude" is independent of the hard process.

A way to see it is to write the internal momenta in terms of an on-shell momenta

$$\bar{q} = \left(x p^+ , \frac{m_e^2 + |p_\perp|^2}{x p^+} , - \vec{p}_\perp \right)$$

plus an off-shell correction,

$$q^\mu = \bar{q}^\mu - \frac{q^2 - m_e^2}{x p^+} n_-^\mu$$

with n_-^μ being $(0, 1, \vec{0})$, the null vector in the $-$ direction.

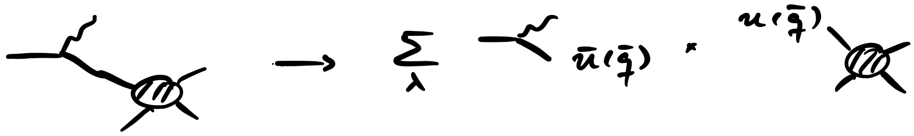
For low virtuality, $q \simeq \bar{q}$. substituting this in the numerator,

$$\not{q} + m_e \simeq \bar{\not{q}} + m_e = \sum_\lambda u_\lambda(\bar{q}) \bar{u}_\lambda(\bar{q})$$

using the completeness relation for on shell momenta.

Then, the propagator splits the

amplitude in two,



In the hard process, $\vec{p}_\perp$ is small and negligible compared with the other scales. So in the hard part,

$$\bar{q}^\mu \to \bar{q}^\mu_{coll} = (x p^+, 0, 0) \qquad \text{if } m_e \text{ also negl.}$$

is a good approx.

- splitting amplitudes

Just to make the calculation easier, we can take $m_e \to 0$.

$$A^{(h)}_{split} = e\, \bar{u}_h(\bar{q}) \gamma^\mu u(p)\, \varepsilon^*_\mu(k)$$

One can compute explicity these splitting amplitudes in terms of x & $p_\perp$,

$$A^{(+)}_{split}(e_+ \to \gamma_+ e_+) = \sqrt{2}\, e\, \underset{\substack{\shortparallel \\ |p_\perp| e^{-i\phi}}}{p_\perp^*} \cdot \frac{1}{\sqrt{x}\,(1-x)}$$

$$A^{(-)}_{split}(e_+ \to \gamma_+ e_-) \propto m_e$$

$$A^{(+)}_{split}(e_+ \to \gamma_- e_+) = \sqrt{2}\, e\, \underset{\substack{\shortparallel \\ |p_\perp| e^{i\phi}}}{p_\perp} \frac{\sqrt{x}}{1-x}$$

$$A^{(-)}_{split}(e_+ \to \gamma_- e_-) \propto m_e$$

With this we can write the cross section for e^+e^- to produce $\mu^+\mu^-$ + a collinear photon which might go unobserved along the beam.

$$\sigma(e^+e^- \to \mu^+\mu^-\gamma) = \frac{1}{2s} \int d\Phi_{\mu\mu} \int \frac{d^3k}{(2\pi)^3 2E_k} |A|^2$$

the photon phase space,

$$\int \frac{d^3k}{(2\pi)^3 2E_k} = \frac{dk^+ \, d^2p_\perp}{(2\pi)^3 \, 2k^+} = \frac{dx \; |p_\perp|^2 \, dp_\perp \, d\phi}{(2\pi)^3 \, 2(1-x)}$$

The splitting amplitude does depend on $p_\perp$, but the hard amplitude doesn't. But it depends on x. So

$$\sigma(e^+e^- \to \mu^+\mu^-\gamma) = \int dx\, f_e(x)\, \hat{\sigma}(e^-e^+ \to \mu^+\mu^-)$$

So the e^- in the hard scattering does not have a fixed energy.

The discussion so far is used to compute only part of the "pdf" f_e, given by the real contributions.

$$f_e(x) = \int \frac{|p_\perp|^2 d|p_\perp|\, d\phi}{(2\pi)^3\, 2(1-x)} \cdot \underbrace{\left(\frac{1}{q^2 - m_e^2}\right)^2}_{\frac{(1-x)^2}{|p_\perp|^4}} \cdot \overset{\text{flux factor}}{x} \cdot 2e^2 |p_\perp|^2 \left(\frac{1}{x(1-x)^2} + \frac{x}{(1-x)^2}\right)$$

$$= \frac{e^2}{8\pi^2} \frac{1+x^2}{1-x} \int_{m_e^2}^{Q^2} \frac{d|p_\perp|^2}{|p_\perp|^2} \int \frac{d\phi}{2\pi}$$

$$= \frac{\alpha}{2\pi} \frac{1+x^2}{1-x} \log \frac{Q^2}{m_e^2}$$

Q^2 is the factorization scale. Is the maximal energy up to which we integrate the radiation.

For $Q^2 \simeq m_Z^2$, $\log \frac{Q^2}{m_e^2} \sim 25$

$Q^2 \simeq (10\,\text{GeV})^2$, $\log \frac{(10\,\text{GeV})^2}{m_e^2} \sim 20$

So the log is a large number, that depends on the kinematical configuration of our scattering.

Perturbation theory is then an expansion in α (hard expansion), but also in $\alpha \cdot \log$.

We still need to compute the virtual contribution to the f_e.

However, this can be fixed by requiring that f_e is a probability distribution, so

$$\int_0^1 dx\, f_e(x) = 1.$$

The virtual correction shifts the $x=1$ point,

$$f_e(x) = f_v\, \delta(1-x) + \frac{\alpha}{2\pi} \frac{1+x^2}{1-x} \log \frac{Q^2}{m_e^2}$$

Therefore,

$$f_v = 1 - \frac{\alpha}{2\pi} \log \frac{Q^2}{m_e^2} \int_0^1 dx\, \frac{1+x^2}{1-x}$$

There is a nice way to write this using the plus distributions. Write

$$1+x^2 = (1-x)^2 - 2(1-x) + 2$$

then

$$\int_0^1 dx\, \frac{1+x^2}{1-x} = \int_0^1 dx\, \frac{2}{1-x} - \frac{3}{2}$$

So we write

$$f_e(x) = \delta(1-x) \left[1 + \frac{\alpha}{2\pi} \log \frac{Q^2}{m_e^2} \times \frac{-3}{2} \right]$$

$$+ \frac{\alpha}{2\pi} \frac{1+x^2}{[1-x]_+} \log \frac{Q^2}{m_e^2}$$

Where the plus distribution is defined as

$$\int_0^1 dx\, g(z) \left[\frac{f(z)}{1-z}\right]_+ = \int_0^1 dx\, f(z)\, \frac{g(z)-g(0)}{1-z}$$

This trick to extract the α correction on α seems this, a trick. However, one could compute it, as it comes from the virtual diagram, as it contributes to $x=1$. If we were to compute it we would get the above.

In dim-reg, the $1/\epsilon$ singularity of the virtual contrib. cancels with the $1/\epsilon$ of the collinear, giving a finite contribution. This cancellation generates the plus distribution & it is guaranteed by the KLN theorem.

Time permits, we'll revisit this explicity for QCD.

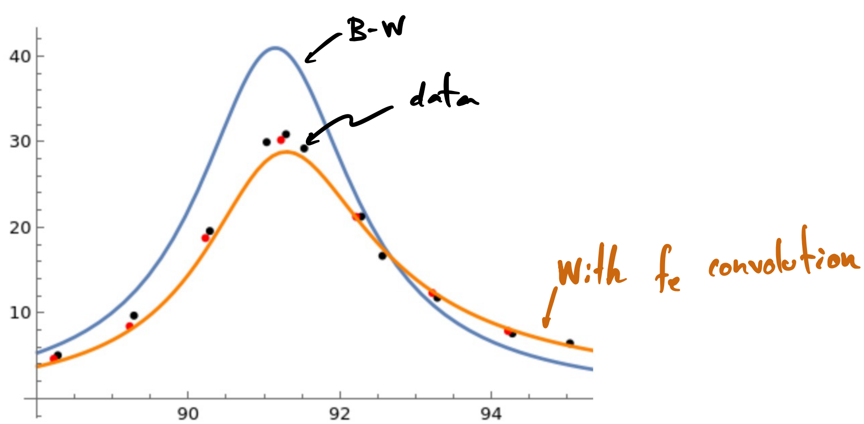
- Using the above expression for f_e, one has

$$\sigma(e^+e^- \to \mu\mu\gamma) = \int dx_1 dx_2 \, f_e^{(e)}(x_1) f_{\bar{e}}^{(\bar{e})}(x_2) \cdot \hat{\sigma}(e^+e^- \to \mu^+\mu^-)$$

with the partonic center of mass energies given by

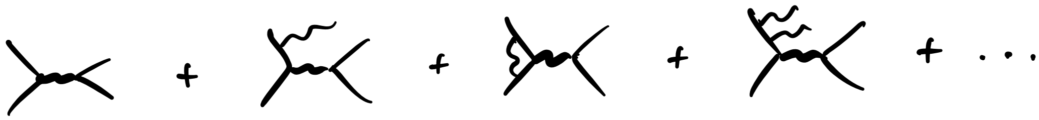
$$\hat{s} = x_1 x_2 s$$

Using this with the partonic cross section we computed before, we get a qualitative matching with data and a much better quantitative comparison.



• DGLAP evolution

We can consider the emission of not only one photon, but multiple



We cannot consider all diagrams, but we can resum the contributions from the ones that are log enhanced.

Before we computed the DGLAP splitting function

$$P_{ee}(z) = \frac{\alpha}{2\pi}\left[(1+z^2)\left[\frac{1}{1-z}\right]_+ + \frac{3}{2}\delta(1-z)\right]$$

after Altarelli, Parisi, and Dokshitzer, Gribov, Lipatov.

Notice that the integral vanishes,

$$\int_0^1 dz \, P_{ee}(z) = 0,$$

as it should be to keep

$$\int_0^1 dx \, f_e(x) = 1.$$

The indices in P_{ee} denote the splitting function of an e^- going to e^- with energy fraction x.

More explicitly, we write the above as $P_{e \leftarrow e}(z)$. At this order, we have

$$P_{\gamma \leftarrow e}(z) = \frac{\alpha}{2\pi} \frac{1 + (1-z)^2}{z}.$$

We add explicitly the Q^2 dependence on the probability distributions,

$$f_e^{(e)}(x, Q^2)$$

denoting the probability to "find" an electron in the hard scattering with energy fraction x, when starting with an electron and integrating over photon emission up to $|p_\perp|^2 = Q^2$.

If $Q^2 \to Q^2 + \Delta Q^2$, we must consider the possibility of emitting an extra photon.

This gives

$$f_e^{(e)}(x, Q+\Delta Q) = f_e^{(e)}(x, Q) +$$

$$+ \int_0^1 dx' \int_0^1 dz \, \frac{\Delta Q^2}{Q^2} P_{e\leftarrow e}(z) \, f_e^{(e)}(x', p_\perp) \, \delta(x - zx')$$

$$= f_e^{(e)}(x, Q) + \frac{\Delta Q^2}{Q^2} \int_x^1 \frac{dz}{z} P_{e\leftarrow e}(z) \, f_e^{(e)}\left(\frac{x}{z}, p_\perp\right)$$

Passing to cont. evolution,

$$\frac{d}{d\ln Q} f_e^{(e)}(x, Q) = \int_x^1 \frac{dz}{z} P_{e\leftarrow e}(z) \, f_e^{(e)}\left(\frac{x}{z}, p_\perp\right)$$

Solving this integro-differential equation allows to resum the radiation of multiple photons from the e^- due to collinear splitting, given that one accompanies it with boundary conditions, i.e.

$$f_e^{(0)}(x, m^2) = \delta(1-x)$$

- Adding the effect of photon splitting,

on top of the collinear radiation, allows to write a closed system of equations for the distributions in QED.

Given

$$P_{e\leftarrow e} = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$$

$$P_{\gamma\leftarrow e} = \frac{1+(1-z)^2}{z}$$

$$P_{e\leftarrow\gamma} = z^2 + (1-z)^2$$

$$P_{\gamma\leftarrow\gamma} = -\frac{2}{3}\delta(1-z)$$

one has

$$\frac{d}{d\ln Q} f_\gamma(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{\gamma\leftarrow e}(z)\Big(f_e\big(\tfrac{x}{z},Q\big) + f_{\bar{e}}\big(\tfrac{x}{z},Q\big)\Big) + P_{\gamma\leftarrow\gamma}(z)\, f_\gamma\big(\tfrac{x}{z},Q\big)\Big]$$

$$\frac{d}{d\ln Q} f_e(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{e\leftarrow e}(z)\, f_e\big(\tfrac{x}{z},Q\big) + P_{e\leftarrow\gamma}(z)\, f_\gamma\big(\tfrac{x}{z},Q\big)\Big]$$

$$\frac{d}{d\ln Q} f_{\bar{e}}(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{e\leftarrow e}(z)\, f_{\bar{e}}\big(\tfrac{x}{z},Q\big) + P_{e\leftarrow\gamma}(z)\, f_\gamma\big(\tfrac{x}{z},Q\big)\Big]$$

with boundary conditions, for the e^- case, given by

$$f_e(x,m^2) = \delta(1-x)$$

$$f_\gamma(x,m^2) = 0$$

$$f_{\bar{e}}(x,m^2) = 0$$

The evolution equations satisfy

$$\int_0^1 dx \left(f_e(x,Q) - f_{\bar{e}}(x,Q) \right) = 1$$

and momentum conservation

$$\int_0^1 dx \; x \left(f_e(x,Q) + f_{\bar{e}}(x,Q) + f_\gamma(x,Q) \right) = 1.$$